DETERMINING THE STEADY-STATE TEMPERATURE FIELD IN A CRACKED PLATE WITH HEAT TRANSFER FROM THE LATERAL SURFACES

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The heat conduction problem is considered in the case of a plate with a thermally insulated straight crack and with a given prior temperature field in the solid plate. The heat transfer from the plate to the ambient medium is assumed to follow Newton's law.

We consider an infinitely large thin plate of thickness 2δ with a thermally insulated straight crack of length $2l$, the latter located on the 0x axis symmetrically about the origin of coordinates. The heat transfer from this plate to the ambient medium is assumed to follow Newton's law, and the temperature field in the solid plate is described by a given function $t^*(x, y)$. It is required to determine the temperature field $T(x, y)$ satisfying both the equation of heat conduction

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \varkappa^2 T = -\varkappa^2 t_c, \tag{1}$$

in dimensionless variables referred to half the crack length l and also the condition of thermal insulation

$$\frac{\partial T}{\partial y} = 0 \qquad \text{for} \quad y = 0, \quad |x| < 1. \tag{2}$$

The general solution to Eq. (1) will be sought in the form

$$T(x, y) = t(x, y) + t^*(x, y), \tag{3}$$

where $t(x, y)$ is the solution to the homogeneous equation corresponding to Eq. (1) with condition (2).

With the temperature field $t(x, y)$ represented as an analog of the logarithmic double-layer potential [1]

$$t(x, y) = \frac{1}{2\pi} \int_{-1}^{1} \gamma(x_*) \frac{d}{dy_*} K_0(\varkappa r)\, dx_*, \tag{4}$$

the condition of thermal insulation (2) yields the following expression for the derivative of density $\gamma(x)$ [2]:

$$\int_{-1}^{1} \gamma'(\xi) K(w)\, d\xi = -\frac{\pi}{\varkappa} f(x), \tag{5}$$

where

$$w = \varkappa(\xi - x); \quad f(x) = 2\frac{\partial t^*(x, 0)}{\partial y},$$
$$K(w) = \int_0^\infty \frac{\sqrt{\eta^2 + 1}}{\eta} \sin \eta w d\eta = K_1(w) + \int K_0(w)\, dw. \tag{6}$$

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No exact method of solving Eq. (5) is known yet. We will, therefore, consider certain approximate methods of solution.

1. We will find a solution to Eq. (5) for small values of $\varkappa$ ($\varkappa < 1$). Using the series expansion of functions $K_0(w)$, $K_1(w)$ for small values of w in [3], we obtain

$$K(w) = \frac{1}{w} + \sum_{n=0}^{\infty} a_n w^{2n+1} + \ln(\xi - x) \sum_{n=0}^{\infty} b_n w^{2n+1},$$

$$a_n = b_n \{s_n - (4n+3)[(2n+1)(2n+2)]^{-1}\}, \tag{7}$$

$$b_n = -[(2^n n!)^2 (2n+1)(2n+2)]^{-1}, \qquad s_n = C + \ln\frac{\varkappa}{2} - \sum_{k=1}^{n} \frac{1}{k}, \tag{8}$$

where C is the Euler constant.

Inserting (7) into (5), we have

$$\int_{-1}^{1} \frac{\gamma'(\xi)}{\xi - x} d\xi = -\pi f(x) - \sum_{n=0}^{\infty} \varkappa^{2n+2} \int_{-1}^{1} \gamma'(\xi)[a_n + b_n \ln|\xi - x|](\xi - x)^{2n+1} d\xi \tag{9}$$

or, after application of the inversion theorem,

$$\gamma'(x) = \frac{1}{\pi\sqrt{1-x^2}} \left\{C_1 + \int_{-1}^{1} \frac{\sqrt{1-\tau^2}}{\tau - x} f(\tau)\, d\tau + \frac{1}{\pi} \sum_{n=0}^{\infty} \varkappa^{2n+2} \int_{-1}^{1} \frac{\sqrt{1-\tau^2}}{\tau - x} d\tau \int_{-1}^{1} \gamma'(\xi)[a_n + b_n \ln|\xi - x|](\xi - \tau)^{2n+1} d\xi. \right. \tag{10}$$

Equation (10) will be solved by the asymptotic method shown in [4, 5]. The solution will be sought in the form of a power series in $\varkappa$:

$$\gamma'(x) = \sum_{n=0}^{\infty} \varkappa^{2n} \gamma_n'(x). \tag{11}$$

Inserting (11) into (10) and equating respective terms of the same power in $\varkappa$, we obtain the following recurrence formulas for $\gamma_n'(x)$:

$$\gamma_0'(x) = \frac{1}{\pi\sqrt{1-x^2}} \left[C_1 + \int_{-1}^{1} \frac{\sqrt{1-\tau^2}}{\tau - x} f(\tau)\, d\tau\right],$$

$$\gamma_{i+1}'(x) = \frac{1}{\pi^2\sqrt{1-x^2}} \left\{\int_{-1}^{1} \frac{\sqrt{1-\tau^2}}{\tau - x} d\tau \int_{-1}^{1} \sum_{k=0}^{i} \gamma_k'(\xi)[a_{i-k} + b_{i-k} \ln|\xi - \tau|](\xi - \tau)^{2(i-k)+1} d\xi\right\} \quad (i = 0, 1, 2, \ldots). \tag{12}$$

Having determined the unknown function $\gamma'(x)$ from (11) with the aid of (12), we then determine $\gamma(x)$ directly by integration. The arbitrary integration constant C_1 is then determined from the condition that $\gamma(\pm 1) = 0$.

2. We will now consider large values of $\varkappa$ ($\varkappa > 1$). By the method developed in [6, 7] we find the asymptotic solution to Eq. (5) as the combination

$$\gamma'(\tau) = \omega[\varkappa(1+\tau)] - \omega[\varkappa(1-\tau)] \tag{13}$$

of solutions $\omega(\tau)$ to the Wiener–Hopf integral equation

$$\int_{0}^{\infty} \omega(\tau) K(\tau - \zeta)\, d\tau = -\pi\psi(\zeta). \tag{14}$$

Here function $\psi(\zeta) = (2/\varkappa)f(\zeta/\varkappa - 1)$ continues analytically into the region $2\varkappa \le \zeta < \infty$.

Equation (14) with kernel (6) is solved exactly by the Weiner–Hopf method. Considering that $\gamma(\pm 1) = 0$ and taking into account expression (13), we determine function $\gamma(x)$ as

$$\gamma(x) = \int_x^1 \{\omega[\varkappa(1+\xi)] - \omega[\varkappa(1-\xi)]\}\, d\xi. \tag{15}$$

We note that for a semiinfinite crack $(0 \le x < \infty)$ Eq. (5) becomes Eq. (14) and, as a consequence, we obtain an exact solution to the problem.

3. An approximate solution to Eq. (5) can be obtained in closed form, if the following approximation† is used in expression (6)

$$\frac{\sqrt{\eta^2+1}}{\eta} \cong \operatorname{cth} \eta. \tag{16}$$

The kernel of the integral equation (5) becomes then

$$K[\varkappa(\xi - x)] \cong \frac{\pi}{2}\left(1 - \frac{2v}{u-v}\right), \tag{17}$$

and Eq. (5) can be written as

$$\int_c^d \frac{\gamma'(u)\, du}{u-v} = -\frac{1}{\varkappa} \cdot \frac{f(v)}{v}, \tag{18}$$

with $u = \exp(\pi\varkappa\xi)$, $v = \exp(\pi\varkappa x)$, $c = \exp(-\pi x)$, and $d = \exp(\pi\varkappa)$.

The solution to Eq. (18) is given by the inversion formula

$$\gamma'(v) = \frac{1}{\pi\varkappa\sqrt{(d-v)(v-c)}}\left\{\frac{1}{\pi}\int_c^d \frac{\sqrt{(d-u)(u-c)}}{u(u-v)} f(u)\, du + C_2\right\}. \tag{19}$$

Integrating Eq. (19) and changing to variable x yields $\gamma(x)$. The arbitrary constant C_2 and the integration constant are determined from the condition that $\gamma(\pm 1) = 0$.

Formula (19) yields an approximate solution to Eq. (5). It can be shown, moreover, that the error of this solution does not exceed the error of that approximation (16).

Thus, with the function $\gamma(x)$ known, formula (4) yields the perturbation temperature $t(x, y)$ and, specifically,

$$t(x, \pm 0) = \pm \frac{1}{2}\gamma(x). \tag{20}$$

Example. Let the temperature of the upper and the lower surface of an infinitely long strip, a plate of width 2b, be $\pm t_0$ and the ambient temperature be zero. Then

$$t^*(y) = \frac{q}{\varkappa} \operatorname{sh} \varkappa y, \tag{21}$$

with $q = \varkappa t_0 \sinh \varkappa b$. If the width b is sufficiently large, then the plate can be considered infinite.

In the plate there is a crack of length $2l$ on the 0x axis, symmetrical about the 0y axis. The 0x axis is equidistant from the upper and the lower strip-plate surface.

In order to determine the perturbation temperature in such a plate, it is necessary to find the function $\gamma(x)$ from Eq. (5), where $f(x) = \partial t^*/\partial y\,|_{y=0} = q$. We will seek a solution to this equation according to the formulas provided here.

1. For small values of $\varkappa$, function $\gamma(x)$ is found by formula (11) in conjunction with (12). Retaining only the first three terms, we have

†The maximum relative error of such an approximation is as high as 8.3% when $\eta = 1.2$ and it decreases fast as η increases or decreases.

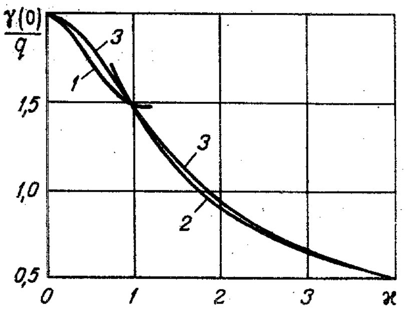

Fig. 1. Graph of $\gamma(0)/q = f(\varkappa)$: 1) according to formula (22); 2) according to formula (24); 3) according to formula (25).

$$\gamma(x) = q \sum_{k=0}^{2} d_{2k+1} (1 - x^2)^{k+\frac{1}{2}},$$

where

$$d_1 = 2[1 + (0.25 \ln \varkappa - 0.3272)\varkappa^2 + (0.0625 \ln^2 \varkappa - 0.1253 \ln \varkappa - 0.0253)\varkappa^4],$$
$$d_3 = -2[0{,}0833\varkappa^2 + (0{,}0313 \ln \varkappa - 0{,}1228)\varkappa^4], \quad d_5 = 0{,}0062\varkappa^4.$$

2. For large values of $\varkappa$, the solution to Eq. (14) is

$$\omega(\tau) = \frac{2q}{\sqrt{\pi}} \cdot \frac{\exp(-\tau)}{\sqrt{\tau}}$$

and then

$$\gamma(x) = \frac{2q}{\varkappa} \{\operatorname{erf} \varphi(x) + \operatorname{erf} \varphi(-x) - \operatorname{erf} \varphi(1)\},$$

where

$$\varphi(x) = \sqrt{\varkappa(1+x)}; \quad \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2)\, dz.$$

3. Finally, formula (19) yields

$$\gamma(x) = \frac{2q}{\pi\varkappa} \sum_{k=1}^{2} \arcsin\{\operatorname{cth} \pi\varkappa - \operatorname{csh} \pi\varkappa \exp[(-1)^k \pi\varkappa x]\}.$$

The quantity $\gamma(0)/q$ has been plotted in Fig. 1 as a function of $\varkappa$: curve 1 represents formula (22) for $\varkappa < 1$, curve 2 represents formula (24) for $\varkappa > 1$, and curve 3 represents formula (25) for all values of $\varkappa$.

It can be seen here that curves 1 and 2 come close together as $\varkappa = 1$ and do not differ much from curve 3. Formula (25) is most convenient for practical calculations, because it applies to all values of $\varkappa$.

NOTATION

$T(x, y)$	is the temperature of plate with crack;
$t_a(x, y)$	is the temperature of ambient medium;
$t^*(x, y)$	is the temperature of plate without a crack;
$t(x, y)$	is the function characterizing the perturbation of the temperature field by the presence of a crack;
$\varkappa^2 = \alpha l/\delta\lambda$;	
α	is the heat transfer coefficient;
$2l$	is the length of crack;
2δ	is the thickness of plate;
λ	is the thermal conductivity;
$K_0(x), K_1(x)$	are the zeroth-order and first-order MacDonald function;
r	is the distance of point $(x, 0)$ $(-1 \le x \le 1)$ from an arbitrary point on the x0y plane;
$\gamma(x)$	is the density of the analog of the logarithmic double-layer potential.

LITERATURE CITED

1. O. V. Poberezhnyi and G. S. Kit, Inzh. Fiz. Zh., 15, No. 4 (1968).
2. G. S. Kit and O. V. Poberezhnyi, Dokl. Akad. Nauk UkrSSR, Ser. A, No. 10 (1970).
3. E. Gray and G. B. Mattews, Bessel Functions and Their Applications in Physics and Mechanics [Russian translation], Izd. IL, Moscow (1949).
4. V. M. Aleksandrov, Izv. Akad. Nauk, SSSR, Mekhan. Tverd. Tela, No. 4 (1967).
5. V. M. Aleksandrov, in: Stress Concentration [in Russian], Ed. 2 (1968).
6. V. M. Aleksandrov and B. I. Smetanin, Prikl. Matem. i Mekhan., 29, No. 4 (1968).
7. V. M. Aleksandrov, Prikl. Matem. i Mekhan., 27, No. 5 (1963).